ON A TWISTED JACQUET MODULE OF $\mathrm{GL}(2n)$ OVER A FINITE FIELD

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Abstract. Let F be a finite field and $G = \mathrm{GL}(2n, F)$. In this paper, we explicitly describe a certain twisted Jacquet module of an irreducible cuspidal representation of G.

1. Introduction

Let F be a finite field and $G = \mathrm{GL}(n, F)$. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let π be any irreducible finite dimensional complex representation of G and ψ be an irreducible representation of N. Let $\pi_{N,\psi}$ be the sum of all irreducible representations of N inside π, on which π acts via the character ψ. It is easy to see that $\pi_{N,\psi}$ is a representation of the subgroup M_ψ of M, consisting of those elements in M which leave the isomorphism class of ψ invariant under the inner conjugation action of M on N. The space $\pi_{N,\psi}$ is called the *twisted Jacquet module* of the representation π. It is an interesting question to understand for which irreducible representations π, the twisted Jacquet module $\pi_{N,\psi}$ is non-zero and to understand its structure as a module for M_ψ.

In [2],[1], inspired by the work of Prasad in [6], we studied the structure of a certain twisted Jacquet module of a cuspidal representation of $\mathrm{GL}(6, F)$ and $\mathrm{GL}(4, F)$. Based on our calculations, we had conjectured the structure of the module for $\mathrm{GL}(2n, F)$ (see Section 1 in [1]). For a more detailed introduction and the motivation to study the problem, we refer the reader to Section 1 in [2].

Before we state our result, we set up some notation. Let F be a finite field and F_n be the unique field extension of F of degree n. Let $G = \mathrm{GL}(2n, F)$ and $P = MN$ be the standard maximal parabolic subgroup of G corresponding to the partition (n, n). We have, $M \simeq \mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$ and $N \simeq \mathrm{M}(n, F)$. We let $\pi = \pi_\theta$ to be an irreducible cuspidal representation of G associated to the regular character θ. Let ψ be any character of $N \simeq \mathrm{M}(n, F)$ and ψ_0 be a fixed non-trivial character of F. We let

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{M}(n, F)$$

Let $\psi_A : N \to \mathbb{C}^\times$ be the character given by

$$\psi_A\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \psi_0(\mathrm{Tr}(AX)).$$

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Let $H_A = M_1 \times M_2$ where M_1 is the Mirabolic subgroup of $\mathrm{GL}(n,F)$ and $M_2 = w_0 M_1^{\top} w_0^{-1}$ where

$$w_0 = \begin{bmatrix} 0 & \dots & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \dots & 0 \end{bmatrix}$$

Let U be the subgroup of unipotent matrices in $\mathrm{GL}(2n,F)$ and $U_A = U \cap H_A$. Then, we get $U_A \simeq U_1 \times U_2$ where U_1 and U_2 are the upper triangular unipotent subgroups of $\mathrm{GL}(n,F)$. For $k = 1,2$, let $\mu_k : U_k \to \mathbb{C}^{\times}$ be the non-degenerate character of U_k given by

$$\mu_k \left(\begin{bmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1,n} \\ & 1 & x_{23} & \cdots & x_{2,n} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & x_{(n-1),n} \\ & & & & 1 \end{bmatrix} \right) = \psi_0 \left(x_{12} + x_{23} + \cdots + x_{(n-1),n} \right).$$

Let $\mu : U_A \to \mathbb{C}^{\times}$ be the character of U_A given by

$$\mu(u) = \mu_1 (u_1) \mu_2 (u_2)$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \in U_A$.

Theorem 1.1. *Let θ be a regular character of $F_{2n}^{\times}$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^{\times}} \otimes \mathrm{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

In the case of $\mathrm{GL}(4,F)$ and $\mathrm{GL}(6,F)$, we did explicit character computations to establish the the structure of the module. However, to imitate these character computations for $\mathrm{GL}(2n,F)$ will be tedious. In this paper, we prove our main theorem by using the "multiplicity one theorem" for $\mathrm{GL}(2n,F)$ and completely avoid character calculations.

Currently, we are investigating the structure of π_{N,ψ_A} in the case when F is a p-adic field. We hope to write up the details at a later time.

2. Preliminaries

In this section, we mention some preliminary results that we need in our paper.

2.1. **Character of a Cuspidal Representation.** Let F be the finite field of order q and $G = \mathrm{GL}(m,F)$. Let F_m be the unique field extension of F of degree m. A character θ of $F_m^{\times}$ is called a "regular" character, if under the action of the Galois group of F_m over F, θ gives rise to m distinct characters of $F_m^{\times}$. It is a well known fact that the cuspidal representations of $\mathrm{GL}(m,F)$ are parametrized by the regular characters of $F_m^{\times}$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used. We refer the reader to Section 6 in [4] for more precise statements on computing character values.

Theorem 2.1. *Let θ be a regular character of $F_m^{\times}$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(m,F)$ *associated to θ. Let Θ_θ be its character. If*

$g \in \mathrm{GL}(m, F)$ is such that the characteristic polynomial of g is not a power of a polynomial irreducible over F. Then, we have

$$\Theta_\theta(g) = 0.$$

Theorem 2.2. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_θ be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element g in $\mathrm{GL}(m, F)$. If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $F_m^\times$. Suppose that s comes from $F_m^\times$. Let z be an eigenvalue of s in F_m and let t be the dimension of the kernel of $g - z$ over F_m. Then*

$$\Theta_\theta(g) = (-1)^{m-1}\Big[\sum_{\alpha=0}^{d-1} \theta(z^{q^\alpha})\Big](1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{t-1}).$$

where q^d is the cardinality of the field generated by z over F, and the summation is over the distinct Galois conjugates of z.

See Theorem 2 in [6] for this version.

2.2. **Kirillov Representation.** Let F be a finite field with q elements and $G = \mathrm{GL}(n, F)$. Let P_n be the Mirabolic subgroup of G and let U be the subgroup of unipotent matrices of G. In this section, we recall the Kirillov representation of the Mirabolic subgroup P_n of G. Let ψ_0 be a non-trivial character of F and let $\psi : U \to \mathbb{C}^\times$ be the non-degenerate character of U given by

$$\psi\left(\begin{bmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ & 1 & x_{23} & \cdots & x_{2n} \\ & & 1 & \cdots & \vdots \\ & & & \ddots & x_{n-1,n} \\ & & & & 1 \end{bmatrix}\right) = \psi_0(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n}).$$

Then, $\mathcal{K} = \mathrm{ind}_U^{P_n} \psi$ is called the *Kirillov representation* of P_n.

Theorem 2.3. *$\mathcal{K} = \mathrm{ind}_U^{P_n} \psi$ is an irreducible representation of P_n.*

We refer the reader to Theorem 5.1 in [3] for a proof.

2.3. **Multiplicity one Theorem for $\mathrm{GL}(n, F)$ over a finite field F.** We continue with the notation of section 2.2.

Theorem 2.4. *Let $\mathcal{G} = \mathrm{ind}_U^G(\psi)$. The representation $\mathcal{G}$ of G is multiplicity free.*

We refer to Theorem 6.1 in [3] for a proof.

2.4. **Twisted Jacquet Module.** In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation π.

Let $G = \mathrm{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of G. Let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \mathrm{Span}_{\mathbb{C}}\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$$

and

$$M_\psi = \{m \in M \mid \psi^m(n) = \psi(n), \forall n \in N\}.$$

Clearly, M_ψ is a subgroup of M and it is easy to see that $V(N, \psi)$ is an M_ψ-invariant subspace of V. Hence, we get a representation $(\pi_{N,\psi}, V/V(N, \psi))$ of M_ψ. We call

$(\pi_{N,\psi}, V/V(N,\psi))$ the twisted Jacquet module of π with respect to ψ. We write $\Theta_{N,\psi}$ for the character of $\pi_{N,\psi}$.

Proposition 2.5. *Let (π, V) be a representation of $\mathrm{GL}(k, F)$ and Θ_π be the character of π. We have*

$$\Theta_{N,\psi}(m) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(mn)\overline{\psi(n)}.$$

We refer the reader to Proposition 2.3 in [2] for a proof.

Remark 2.6. Taking $m = 1$, we get the dimension of $\pi_{N,\psi}$. To be precise, we have

$$\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|} \sum_{n \in N} \Theta_\pi(n)\overline{\psi(n)}.$$

2.5. **q-Hypergeometric Identity.** In this section, we record a certain q-identity from [5] which we use in calculating the dimension of the twisted Jacquet module. Before we state it, we set up some notation. Let $\mathrm{M}(n, m, r, q)$ be the set of all $n \times m$ matrices of rank r over the finite field F of order q and $(a; q)_n$ be the q-Pochhammer symbol defined by

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

Proposition 2.7. *Let a be an integer greater than or equal to $2n$. Then*

$$\sum_{r \geq 0} \mathrm{M}(n, n, r, q)(q; q)_{a-r} = q^{n^2} \frac{(q; q)_{a-n}^2}{(q; q)_{a-2n}}.$$

We refer the reader to Lemma 2.1 in [5] for a proof of the above proposition in a more general set up.

3. Dimension of the Twisted Jacquet Module

Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of G corresponding to the regular character θ of $F_{2n}^\times$ and Θ_θ be its character. In this section, we calculate the dimension of π_{N,ψ_A}, where

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Throughout, we write $\mathrm{M}(n, m, r, q)$ denote the set of $n \times m$ matrices of rank r over the finite field F of cardinality q. For $\alpha \in F$ and $0 \leq r \leq n$, consider the subset $Y_{n,r}^\alpha$ of $\mathrm{M}(n, F)$ given by

$$Y_{n,r}^\alpha = \{X \in \mathrm{M}(n, F) \mid \mathrm{Rank}(X) = r, \mathrm{Tr}(AX) = \alpha\}.$$

Lemma 3.1. *We have*

$$|\,\mathrm{M}(n, n, r, q)| = q^r |\,\mathrm{M}(n, n-1, r, q)| + (q^n - q^{r-1})|\,\mathrm{M}(n, n-1, r-1, q)|.$$

Proof. Let $S = q^r |\,\mathrm{M}(n, n-1, r, q)| + (q^n - q^{r-1})|\,\mathrm{M}(n, n-1, r-1, q)|$. It is well known that

$$|\,\mathrm{M}(n, m, r, q)| = \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^m - q^j)}{(q^r - q^j)}.$$

Thus, we have

$$S = q^r \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^{n-1} - q^j)}{(q^r - q^j)} + (q^n - q^{r-1}) \prod_{j=0}^{r-2} \frac{(q^n - q^j)(q^{n-1} - q^j)}{(q^{r-1} - q^j)}$$

$$= \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^n - q^{j+1})}{(q^r - q^j)} + (q^n - q^{r-1}) \prod_{j=0}^{r-2} \frac{(q^n - q^j)(q^n - q^{j+1})}{(q^r - q^{j+1})}$$

$$= \frac{q^n - q^r}{q^n - 1} |\,\mathrm{M}(n,n,r,q)| + \frac{(q^n - q^{r-1})(q^r - 1)}{(q^n - q^{r-1})(q^n - 1)} |\,\mathrm{M}(n,n,r,q)|$$

$$= |\,\mathrm{M}(n,n,r,q)|.$$

□

Lemma 3.2. *Let $r \in \{1,2,3,\ldots,n\}$ and $\alpha, \beta \in F^\times$. Then we have*

$$|Y_{n,r}^{\alpha}| = |Y_{n,r}^{\beta}|.$$

Proof. Consider the map $\phi : Y_{n,r}^{\alpha} \to Y_{n,r}^{\beta}$ given by

$$\phi(X) = \alpha^{-1}\beta X.$$

Suppose that $\phi(X) = \phi(Y)$. Since $\alpha^{-1}\beta \neq 0$, it follows that ϕ is injective. For $Y \in Y_{n,r}^{\beta}$, let $X = \alpha\beta^{-1}Y$. Clearly, we have $\mathrm{Tr}(AX) = \alpha$ and $\mathrm{Rank}(X) = \mathrm{Rank}(Y) = r$. Thus ϕ is surjective and hence the result. □

Lemma 3.3.
$|Y_{n,r}^{0}| = q^{-1}|\,\mathrm{M}(n,n,r,q)| + (q^r - q^{r-1})|\,\mathrm{M}(n-1,n-1,r,q)| + (q^{r-2} - q^{r-1})|\,\mathrm{M}(n-1,n-1,r-1,q)|.$

Proof. Let $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$ be a basis of F^n over F and $X \in Y_{n,r}^{0}$. Then,

$$[X]_{\mathfrak{B}} = \begin{bmatrix} 0 & w \\ v & Y \end{bmatrix}$$

where w is an $1 \times (n-1)$ row vector, v is an $(n-1) \times 1$ column vector and Y is an $(n-1) \times (n-1)$ block matrix. We also write

$$\begin{bmatrix} w \\ Y \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}$$

where v_i is an $n \times 1$ column vector for $1 \leq i \leq n-1$.

Let V be the $n-1$ dimensional hyperplane spanned by the vectors $\{e_2, e_3, \ldots, e_n\}$. It is easy to see that $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V$. We let W be the space spanned by the vectors $\{v_1, v_2, \ldots, v_{n-1}\}$. Since $X \in Y_{n,r}^{0}$, the rank of the $n \times (n-1)$ matrix

$$\begin{bmatrix} w \\ Y \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}$$

has only two possibilities, either r or $r-1$. We consider both these cases separately.

Case 1) Suppose that

$$\mathrm{Rank}\left(\begin{bmatrix} w \\ Y \end{bmatrix}\right) = \mathrm{Rank}(\begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}) = r.$$

Then $\dim W = r$. It follows that, $\begin{bmatrix} 0 \\ v \end{bmatrix} \in W$ and hence $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W$. Therefore, the number of choices for $\begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}$ is $|\,\mathrm{M}(n, n-1, r, q)|$.

a) If $w = 0$, then

$$W \subseteq V.$$

Hence, $V \cap W = W$ and $\dim(V \cap W) = \dim W = r$. Since $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W$, the number of possibilities of $\begin{bmatrix} 0 \\ v \end{bmatrix}$ will be q^r. Also, the total number of matrices $\begin{bmatrix} w \\ Y \end{bmatrix}$ with rank r and $w = 0$ is $|\,\mathrm{M}(n-1, n-1, r, q)|$.

b) If $w \neq 0$, we have $W \nsubseteq V$. Therefore,

$$\begin{aligned} \dim(W \cap V) &= \dim V + \dim W - \dim(V + W) \\ &= n - 1 + r - n = r - 1. \end{aligned}$$

Since $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V \cap W$, the number of possibilities of $\begin{bmatrix} 0 \\ v \end{bmatrix}$ will be q^{r-1}. The number of matrices $\begin{bmatrix} w \\ Y \end{bmatrix}$ with rank r and $w \neq 0$, is

$$|\,\mathrm{M}(n, n-1, r, q)| - |\,\mathrm{M}(n-1, n-1, r, q)|.$$

Case 2) Suppose that

$$\mathrm{Rank}\left(\begin{bmatrix} w \\ Y \end{bmatrix} \right) = \mathrm{Rank}(\begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix}) = r - 1.$$

Then $\dim W = r - 1$. Therefore, $v \notin W$ and hence $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V \backslash W$. Also,we have that the total number of matrices $\begin{bmatrix} w \\ Y \end{bmatrix}$ with rank $r - 1$ is $|\,\mathrm{M}(n, n-1, r-1, q)|$.

a) If $w = 0$, then $W \subseteq V$. Therefore, $V \cap W = W$ and $\dim(V \cap W) = \dim W = r - 1$. Since $\begin{bmatrix} 0 \\ v \end{bmatrix} \in V \backslash W$, the number of possibilities of $\begin{bmatrix} 0 \\ v \end{bmatrix}$ will be $q^{n-1} - q^{r-1}$. Furthermore, The total number of matrices $\begin{bmatrix} w \\ Y \end{bmatrix}$ with rank $r - 1$ and $w = 0$ is $|\,\mathrm{M}(n-1, n-1, r-1, q)|$.

b) If $w \neq 0$, then $W \nsubseteq V$. Therefore,

$$\begin{aligned} \dim(W \cap V) &= \dim V + \dim W - \dim(V + W) \\ &= n - 1 + r - 1 - n = r - 2. \end{aligned}$$

Since $v \in V \backslash W$, the number of possibilities of $\begin{bmatrix} 0 \\ v \end{bmatrix}$ will be $q^{n-1} - q^{r-2}$. The total number of matrices in this case will be $|\,\mathrm{M}(n, n-1, r-1, q)| - |\,\mathrm{M}(n-1, n-1, r-1, q)|$.

Using Lemma 3.1, and the above computations, we have

$$\begin{aligned}
|Y^0_{n,r}| &= q^r|\operatorname{M}(n-1,n-1,r,q)| + q^{r-1}(|\operatorname{M}(n,n-1,r,q)| - |\operatorname{M}(n-1,n-1,r,q)|) \\
&+ (q^{n-1}-q^{r-1})|\operatorname{M}(n-1,n-1,r-1,q)| \\
&+ (q^{n-1}-q^{r-2})(|\operatorname{M}(n,n-1,r-1,q)| - |\operatorname{M}(n-1,n-1,r-1,q)|) \\[1em]
&= q^{r-1}|\operatorname{M}(n,n-1,r,q)| + (q^{n-1}-q^{r-2})|\operatorname{M}(n,n-1,r-1,q)| \\
&+ (q^r-q^{r-1})|\operatorname{M}(n-1,n-1,r,q)| \\
&+ (q^{n-1}-q^{r-1}-q^{n-1}+q^{r-2})|\operatorname{M}(n-1,n-1,r-1,q)| \\[1em]
&= q^{r-1}|\operatorname{M}(n,n-1,r,q)| + (q^{n-1}-q^{r-2})|\operatorname{M}(n,n-1,r-1,q)| \\
&+ (q^r-q^{r-1})|\operatorname{M}(n-1,n-1,r,q)| + (q^{r-2}-q^{r-1})|\operatorname{M}(n-1,n-1,r-1,q)| \\[1em]
&= q^{-1}|\operatorname{M}(n,n,r,q)| + (q^r-q^{r-1})|\operatorname{M}(n-1,n-1,r,q)| \\
&+ (q^{r-2}-q^{r-1})|\operatorname{M}(n-1,n-1,r-1,q)|.
\end{aligned}$$

□

Lemma 3.4. *We have*
$|Y^1_{n,r}| = q^{-1}|\operatorname{M}(n,n,r,q)| - q^{r-1}|\operatorname{M}(n-1,n-1,r,q)| + q^{r-2}|\operatorname{M}(n-1,n-1,r-1,q)|.$

Proof. Using Lemma 3.2, we have

$$|Y^0_{n,r}| + (q-1)|Y^1_{n,r}| = |\operatorname{M}(n,n,r,q)|.$$

Thus we get,

$$\begin{aligned}
|Y^1_{n,r}| &= \frac{\operatorname{M}(n,n,r,q)| - |Y^0_{n,r}|}{q-1} \\[1em]
&= \frac{\operatorname{M}(n,n,r,q)| - q^{-1}|\operatorname{M}(n,n,r,q)|}{q-1} \\
&- \frac{(q^r-q^{r-1})|\operatorname{M}(n-1,n-1,r,q)| + (q^{r-2}-q^{r-1})|\operatorname{M}(n-1,n-1,r-1,q)|}{q-1} \\[1em]
&= q^{-1}|\operatorname{M}(n,n,r,q)| - q^{r-1}|\operatorname{M}(n-1,n-1,r,q)| + q^{r-2}|\operatorname{M}(n-1,n-1,r-1,q)|.
\end{aligned}$$

□

Lemma 3.5. *We have*

$$|Y^0_{n,r}| - |Y^1_{n,r}| = q^r|\operatorname{M}(n-1,n-1,r,q)| - q^{r-1}|\operatorname{M}(n-1,n-1,r-1,q)|.$$

Proof. Follows from Lemma 3.3 and Lemma 3.4. □

Lemma 3.6. *Let $r \in \{0,1,2,\ldots,n\}$ and $X \in \operatorname{M}(n,n,r,q)$. We have*

$$\Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \begin{cases} (-1)(q;q)_{2n-1}, & \text{if } r = 0 \\ (-1)(q;q)_{2n-2}, & \text{if } r = 1 \\ \vdots & \\ (-1)(q;q)_{n-1}, & \text{if } r = n \end{cases}$$

Proof. The proof follows from Theorem 2.2 above and rewriting the character values using the q-Pochhammer symbol. □

Theorem 3.7. *Let θ be a regular character of $F_{2n}^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(2n, F)$*. We have*

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (q-1)^2(q^2-1)^2 \cdots (q^{n-1}-1)^2 = (q;q)_{n-1}^2.$$

Proof. It is easy to see that the dimension of π_{N,ψ_A} is given by

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = \frac{1}{q^{n^2}} \sum_{X \in \mathrm{M}(n,F)} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))}.$$

Clearly, we have $\mathrm{M}(n,F) = \bigcup_{r=0}^{n} \left(\bigcup_{\alpha \in F} Y_{n,r}^\alpha \right)$. Using this, we see that

$$\begin{aligned}
\dim_{\mathbb{C}}(\pi_{N,\psi_A}) &= \frac{1}{q^{n^2}} \sum_{r=0}^{n} \sum_{\substack{X \in Y_{n,r}^\alpha \\ \alpha \in F}} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\alpha)} \\
&= \frac{1}{q^{n^2}} \sum_{r=0}^{n} (-1)(q;q)_{2n-1-r} \left(|Y_{n,r}^0| - |Y_{n,r}^1| \right) \\
&= -\frac{1}{q^{n^2}} \big[(q;q)_{2n-1} q^0 \left|\mathrm{M}(n-1,n-1,0,q)\right| \\
&+ \sum_{r=1}^{n} (q;q)_{2n-1-r} (q^r \left|\mathrm{M}(n-1,n-1,r,q)\right| - q^{r-1} \left|\mathrm{M}(n-1,n-1,r-1,q)\right|) \big] \\
&= -\frac{1}{q^{n^2}} \sum_{r=0}^{n-1} q^r \left((q;q)_{2n-1-r} - (q;q)_{2n-2-r} \right) \left|\mathrm{M}(n-1,n-1,r,q)\right| \\
&= \frac{1}{q^{n^2}} \sum_{r=0}^{n-1} q^{2n-1} \left|\mathrm{M}(n-1,n-1,r,q)\right| (q;q)_{2n-2-r} \\
&= \frac{1}{q^{(n-1)^2}} \sum_{r=0}^{n-1} \left|\mathrm{M}(n-1,n-1,r,q)\right| (q;q)_{2n-2-r} \\
&= (q;q)_{n-1}^2.
\end{aligned}$$

□

Remark 3.8. Suppose that $B = Aw_0$. It is easy to see that $\Theta_{N,\psi_A}\left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right) = \Theta_{N,\psi_B}\left(\begin{bmatrix} w_0 m_1 w_0 & 0 \\ 0 & m_2 \end{bmatrix}\right)$. Thus we have that $\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = \dim_{\mathbb{C}}(\pi_{N,\psi_B})$.

4. Main Theorem

In this section, we prove the main result of this paper. Before we continue, we set up some notation and record a few preliminary results that we need. Let $G = \mathrm{GL}(2n, F)$ and P be the maximal parabolic subgroup of G with Levi decomposition $P = MN$, where $M \simeq \mathrm{GL}(n,F) \times \mathrm{GL}(n,F)$ and $N \simeq \mathrm{M}(n,F)$. We write F_n for the unique field extension of F of degree n. Let ψ_0 be a fixed non-trivial additive character of F. Let

$$A = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $\psi_A : N \to \mathbb{C}^\times$ be the character of N given by

$$\psi_A\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \psi_0(\mathrm{Tr}(AX)).$$

Let $H_A = M_1 \times M_2$ where M_1 is the Mirabolic subgroup of $\mathrm{GL}(n, F)$ and $M_2 = w_0 {M_1}^\top {w_0}^{-1}$. Let U be the subgroup of unipotent matrices in $\mathrm{GL}(2n, F)$. Let $U_A = H_A \cap U$. Clearly, we have $U_A \simeq U_1 \times U_2$ where U_1 and U_2 are the upper triangular unipotent subgroups of $\mathrm{GL}(n, F)$. For $k = 1, 2$, let $\mu_k : U_k \to \mathbb{C}^\times$ be the non-degenerate character of U_k given by

$$\mu_k\left(\begin{bmatrix} 1 & x_{12} & x_{13} & \dots & x_{1n} \\ & 1 & x_{23} & \dots & x_{2n} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & x_{(n-1)n} \\ & & & & 1 \end{bmatrix}\right) = \psi_0(x_{12} + x_{23} + \cdots + x_{(n-1)n}).$$

Let $\mu : U_A \to \mathbb{C}^\times$ be the character of U_A given by

$$\mu(u) = \mu_1(u_1)\mu_2(u_2)$$

where $u = \begin{bmatrix} u_1 & \\ & u_2 \end{bmatrix}$.

Lemma 4.1. *Let $M_{\psi_A} = \{m \in M \mid \psi_A^m(n) = \psi_A(n), \forall n \in N\}$. Then we have*

$$M_{\psi_A} = \left\{ \begin{bmatrix} C & x & & \\ 0 & a & & \\ & & a & y \\ & & 0 & D \end{bmatrix} \middle| \, a \in F^\times, C, D \in \mathrm{GL}(n-1, F), x, y \in F^{n-1} \right\}.$$

Proof. Let $g = \begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix} \in M$. Then $g \in M_{\psi_A}$ if and only if $Ag_1 = g_2 A$. It follows that $g \in M_{\psi_A}$ if and only if $g_1 = \begin{bmatrix} C & x \\ 0 & a \end{bmatrix}$ and $g_2 = \begin{bmatrix} a & y \\ 0 & D \end{bmatrix}$. □

Lemma 4.2. *Let Z be the center of $G = \mathrm{GL}(2n, F)$. Let H_A be a subgroup of G as above. Then,*

$$M_{\psi_A} \simeq Z \times H_A.$$

Proof. Trivial. □

Lemma 4.3. *Let $\rho_1 = \mathrm{ind}_{U_1}^{M_1} \mu_1$ and $\rho_2 = \mathrm{ind}_{U_2}^{M_2} \mu_2$. Consider the representation (ρ, V) of M_{ψ_A} given by*

$$\rho = \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A} \mu = \theta|_{F^\times} \otimes (\rho_1 \otimes \rho_2).$$

Then (ρ, V) is an irreducible representation of M_{ψ_A}.

Proof. Since ρ_1 is the Kirillov representation of the Mirabolic subgroup M_1 of $\mathrm{GL}(n, F)$, we have that ρ_1 is irreducible (see Theorem 2.3). In a similar way, we can see that ρ_2 is also irreducible. Hence the result. □

Lemma 4.4. *Let $P_{\psi_A} = M_{\psi_A} N$. Consider the map $\tilde{\rho} : P_{\psi_A} \to \mathrm{GL}(V)$ given by*

$$\tilde{\rho}(p) = \tilde{\rho}(mn) = \psi_A(mnm^{-1})\rho(m),$$

where $m \in M_{\psi_A}, n \in N$. Then $(\tilde{\rho}, V)$ is a representation of P_{ψ_A}.

Proof. Let $p_1 = m_1n_1, p_2 = m_2n_2 \in P_{\psi_A}$. Then, we have

$$\begin{aligned}
\tilde{\rho}(p_1p_2) &= \tilde{\rho}(m_1n_1m_2n_2)\\
&= \tilde{\rho}(m_1m_2({m_2}^{-1}n_1m_2)n_2)\\
&= \psi_A(m_1m_2({m_2}^{-1}n_1m_2n_2){m_2}^{-1}{m_1}^{-1})\rho(m_1m_2)\\
&= \psi_A(n_1(m_2n_2{m_2}^{-1}))\rho(m_1m_2)\\
&= \psi_A(n_1)\psi_A(m_2n_2{m_2}^{-1})\rho(m_1)\rho(m_2)\\
&= \psi_A(m_1n_1{m_1}^{-1})\rho(m_1)\psi_A(m_2n_2{m_2}^{-1})\rho(m_2)\\
&= \tilde{\rho}(p_1)\tilde{\rho}(p_2).
\end{aligned}$$

□

Lemma 4.5. *Let $(\tilde{\rho}, V)$ be the representation of P_{ψ_A} given by*

$$\tilde{\rho}(p) = \tilde{\rho}(mn) = \psi_A(mnm^{-1})\rho(m),$$

where $m \in M_{\psi_A}, n \in N$. Then, $(\tilde{\rho}, V)$ is irreducible.

Proof. Let W be a non-trivial P_{ψ_A}-invariant subspace of V. For $w \in W, p \in P_{\psi_A}$, we have

$$\tilde{\rho}(p)w = \psi_A(mnm^{-1})\rho(m)w \in W.$$

Therefore $\rho(m)w \in W$, for all $m \in M_{\psi_A}$, $w \in W$. Since ρ is irreducible (see Lemma 4.3), the result follows. □

Lemma 4.6. *Consider the representation $\tilde{\rho}$ of P_{ψ_A}. We have*

$$\tilde{\rho}|_U = \psi_A \otimes \rho|_{U_A}.$$

Proof. Clearly we have $U = U_AN$. Hence for $u = xn \in U$, we have

$$\tilde{\rho}(u) = \psi_A(xnx^{-1})\rho(x) = \psi_A(n)\rho(x).$$

□

Lemma 4.7. *Let $\rho = \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu$ be the representation of M_{ψ_A} and $\tilde{\rho}$ be the corresponding representation of P_{ψ_A}. For any $z \in Z$, we have*

$$\omega_{\tilde{\rho}}(z) = \omega_\rho(z) = \theta(z).$$

Proof. For $z \in Z$, we have

$$\begin{aligned}
\chi_{\tilde{\rho}}(z) &= \operatorname{Tr}(\tilde{\rho}(z))\\
&= \omega_{\tilde{\rho}}(z)\deg(\rho)\\
&= \operatorname{Tr}(\rho(z))\\
&= \omega_\rho(z)\deg(\rho)\\
&= \operatorname{Tr}(\theta|_{F^\times}(z) \otimes \operatorname{ind}_{U_A}^{H_A} \mu(1))\\
&= \theta(z)\deg(\rho)
\end{aligned}$$

It follows that $\omega_{\tilde{\rho}}(z) = \omega_\rho(z) = \theta(z)$. □

Lemma 4.8. *Let $\chi : F^\times \to \mathbb{C}^\times$ be a character of $F^\times$. Consider the representation $(\tilde{\rho}, V)$ of P_{ψ_A} defined above. Let $\sigma_\chi : P_{\psi_A} \to \mathrm{GL}(V)$ be the map*

$$\sigma_\chi(p) = \sigma_\chi(zhn) = \chi(z)\tilde{\rho}(hn),$$

where $z \in Z, h \in H_A, n \in N$. Then σ_χ is an irreducible representation of P_{ψ_A}.

Proof. It is easy to see that σ_χ is a representation of P_{ψ_A}. Let W be a non-trivial subspace of V invariant under P_{ψ_A} and let $w \neq 0 \in W$. We have

$$\sigma_\chi(zhn)w = \chi(z)\tilde{\rho}(hn)w \in W.$$

Therefore,

$$\tilde{\rho}(zhn)w = \tilde{\rho}(z)\tilde{\rho}(hn)w = \omega_{\tilde{\rho}}(z)\tilde{\rho}(hn)w \in W.$$

Since $\tilde{\rho}$ is irreducible, it follows that $V = W$ and hence the result. □

Lemma 4.9. *Let $\chi_1, \chi_2 \in \widehat{F^\times}$ such that $\chi_1 \neq \chi_2$. Then,*

$$\sigma_{\chi_1} \not\simeq \sigma_{\chi_2}.$$

Proof. Let $z_0 \in Z$ such that $\chi_1(z_0) \neq \chi_2(z_0)$. Let $\chi_{\sigma_{\chi_1}}$, $\chi_{\sigma_{\chi_2}}$ be the characters of σ_{χ_1} and σ_{χ_2}. Suppose that $\sigma_{\chi_1} \simeq \sigma_{\chi_2}$. We have

$$\begin{aligned}
\chi_{\sigma_{\chi_1}}(z_0) &= \mathrm{Tr}(\sigma_{\chi_1}(z_0)) \\
&= \chi_1(z_0)\deg(\rho) \\
&= \chi_{\sigma_{\chi_2}}(z_0) \\
&= \mathrm{Tr}(\sigma_{\chi_2}(z_0)) \\
&= \chi_2(z_0)\deg(\rho).
\end{aligned}$$

The result follows. □

Lemma 4.10. *For $\chi \in \widehat{F^\times}$, we have*

$$\mathrm{Hom}_{P_{\psi_A}}(\sigma_\chi, \mathrm{ind}_U^{P_{\psi_A}} \psi) \neq 0.$$

Proof. Using Fröbenius Reciprocity, we have

$$\mathrm{Hom}_{P_{\psi_A}}(\sigma_\chi, \mathrm{ind}_U^{P_{\psi_A}} \psi) = \mathrm{Hom}_U(\sigma_\chi|_U, \psi).$$

Thus it is enough to show that $\mathrm{Hom}_U(\sigma_\chi|_U, \psi) \neq 0$. For $u \in U$, we have

$$\sigma_\chi|_U(u) = \sigma_\chi(u) = \chi(1)\tilde{\rho}(u) = \tilde{\rho}|_U(u).$$

Therefore,

$$\begin{aligned}
\mathrm{Hom}_U(\sigma_\chi|_U, \psi) &= \mathrm{Hom}_U(\tilde{\rho}|_U, \psi) \\
&= \mathrm{Hom}_U(\psi_A \otimes \rho|_{U_A}, \psi) \\
&= \mathrm{Hom}_U\left(\psi_A \otimes \bigoplus_{s \in U_A \backslash H_A / U_A} \mathrm{ind}_{s^{-1}U_A s \cap U_A}^{U_A} \mu^s, \psi\right) \\
&= \mathrm{Hom}_U(\psi_A \otimes \mu, \psi) \oplus \bigoplus_{1 \neq s \in U_A \backslash H_A / U_A} \mathrm{Hom}_U\left(\psi_A \otimes \mathrm{ind}_{s^{-1}U_A s \cap U_A}^{U_A} \mu^s, \psi\right) \\
&= \mathrm{Hom}_U(\psi, \psi) \oplus \bigoplus_{1 \neq s \in U_A \backslash H_A / U_A} \mathrm{Hom}_U\left(\psi_A \otimes \mathrm{ind}_{s^{-1}U_A s \cap U_A}^{U_A} \mu^s, \psi\right) \\
&\neq 0.
\end{aligned}$$

□

Lemma 4.11. *Let $\chi \in \widehat{F^\times}$ and σ_χ be the irreducible representation of P_{ψ_A}. Then*

$$\mathrm{ind}_U^{P_{\psi_A}} \psi = \bigoplus_{\chi \in \widehat{F^\times}} \sigma_\chi.$$

Proof. The result clearly follows from a simple application of Lemma 4.9 and Lemma 4.10, and computing the degree of $\mathrm{ind}_U^{P_{\psi_A}}(\psi)$. To be precise, suppose that

$$\mathrm{ind}_U^{P_{\psi_A}}(\psi) = (\bigoplus_{x \in \widehat{F^\times}} d_\chi \sigma_\chi) \oplus d\sigma$$

where $d_\chi \geq 1, d \geq 0$ and σ is some representation of P_{ψ_A}. By degree comparison, we have that

$$\deg\big(\bigoplus_{\chi \in \widehat{F^\times}} d_\chi \sigma_\chi\big) = \sum_{\chi \in \widehat{F^\times}} d_\chi \deg(\sigma_\chi) = \sum_{\chi \in \widehat{F^\times}} d_\chi \deg(\rho)$$

Clearly

$$\sum_{\chi \in \widehat{F^\times}} d_\chi \deg(\rho) \geq (q-1)\deg(\rho) = \deg(\mathrm{ind}_U^{P_{\psi_A}}(\psi)),$$

On the other hand, we have

$$\deg(\bigoplus_{\chi \in \widehat{F^\times}} d_\chi \sigma_\chi) + d\deg(\sigma) = \deg(\mathrm{ind}_U^{P_{\psi_A}}(\psi)).$$

It follows that

$$d = 0, d_\chi = 1, \forall \chi \in F^\times.$$

Hence the result. ☐

Lemma 4.12. *Let $m = ah \in M_{\psi_A}$, where $a \in Z$ and $h \in H_A$. Then,*

$$\Theta_{N,\psi_A}(m) = \theta(a)\Theta_{N,\psi_A}(h).$$

Proof. We have

$$\begin{aligned}
\Theta_{N,\psi_A}(m) &= \Theta_{N,\psi_A}(ah)\\
&= \frac{1}{|N|}\sum_{n\in N}\Theta_\theta(ahn)\overline{\psi_A(n)}\\
&= \frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(ahn))\overline{\psi_A(n)}\\
&= \frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(a)\pi(hn))\overline{\psi_A(n)}\\
&= \omega_\pi(a)\frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(hn))\overline{\psi_A(n)}\\
&= \omega_\pi(a)\Theta_{N,\psi_A}(h)
\end{aligned}$$

where ω_π is the central character of π. Explicitly, we have

$$\Theta_\theta(a) = \mathrm{Tr}(\pi(a)) = \mathrm{Tr}(\omega_\pi(a)) = \omega_\pi(a)\dim(\pi).$$

Using Theorem 2.2, it is easy to see that

$$\Theta_\theta(a) = \theta(a)\dim(\pi).$$

Thus, we have $\omega_\pi(a) = \theta(a)$ and the result follows. ☐

Lemma 4.13. *Let $\chi \neq \theta \in \widehat{F^\times}$. Then*

$$\mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) = 0.$$

Proof. It is enough to show that $\dim_{\mathbb{C}} \operatorname{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) = 0$. Clearly, we have

$$\begin{aligned}
\dim_{\mathbb{C}} \operatorname{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) &= \langle \chi_{\pi|_{P_{\psi_A}}}, \chi_{\sigma_\chi} \rangle \\
&= \sum_{zhn \in P_{\psi_A}} \chi_\pi(zhn)\overline{\chi_{\sigma_\chi}(zhn)} \\
&= \sum_{hn \in H_A N} \sum_{z \in Z} \omega_\pi(z)\chi_\pi(hn)\overline{\chi(z)\chi_{\tilde{\rho}}(hn)} \\
&= \sum_{hn \in H_A N} \sum_{z \in Z} \theta(z)\overline{\chi(z)}\chi_\pi(hn)\overline{\chi_{\tilde{\rho}}(hn)} \\
&= \sum_{z \in Z} \theta(z)\overline{\chi(z)} \sum_{hn \in H_A N} \chi_\pi(hn)\overline{\chi_{\tilde{\rho}}(hn)} \\
&= \langle \theta, \chi \rangle \sum_{hn \in H_A N} \chi_\pi(hn)\overline{\chi_{\tilde{\rho}}(hn)} \\
&= 0
\end{aligned}$$

It follows that

$$\operatorname{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) = 0, \forall \chi \in \widehat{F^\times}, \chi \neq \theta.$$

□

Lemma 4.14. *Consider the restriction $\theta|_{F^\times}$ of the regular character θ. Then*

$$\sigma_\theta = \tilde{\rho}$$

as P_{ψ_A} representations.

Proof. Using Lemma 4.7 we have $\omega_{\tilde{\rho}}(z) = \theta(z)$. Thus for $p = zhn \in P_{\psi_A}$, we have

$$\begin{aligned}
\sigma_\theta(zhn) &= \theta(z)\rho(hn) \\
&= \omega_{\tilde{\rho}}(z)\tilde{\rho}(hn) \\
&= \tilde{\rho}(zhn).
\end{aligned}$$

□

4.1. **Proof of the Main Theorem.** For the sake of completeness, we recall the statement below.

Theorem 4.15. *Let θ be a regular character of $F_{2n}^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

Proof. Using transitivity of induction and Lemma 4.11, we have that

$$\begin{aligned}
\mathrm{Hom}_G(\pi, \mathrm{ind}_U^G \psi) &= \mathrm{Hom}_G(\pi, \mathrm{ind}_{P_{\psi_A}}^G (\mathrm{ind}_U^{P_{\psi_A}} \psi)) \\
&= \mathrm{Hom}_G(\pi, \mathrm{ind}_{P_{\psi_A}}^G (\bigoplus_{\chi \in \widehat{F^\times}} \sigma_\chi)) \\
&= \bigoplus_{\chi \in \widehat{F^\times}} \mathrm{Hom}_G(\pi, \mathrm{ind}_{P_{\psi_A}}^G \sigma_\chi) \\
&= \bigoplus_{\chi \in \widehat{F^\times}} \mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) \\
&= \mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\theta) \oplus \bigoplus_{\theta \neq \chi \in \widehat{F^\times}} \mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \sigma_\chi) \\
&= \mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \tilde{\rho})
\end{aligned}$$

Hence,

$$\mathrm{Hom}_G(\pi, \mathrm{ind}_U^G(\psi)) = \mathrm{Hom}_{P_{\psi_A}}(\pi|_{P_{\psi_A}}, \tilde{\rho}) \simeq \mathrm{Hom}_G(\pi, \mathrm{ind}_{P_{\psi_A}}^G \tilde{\rho}) \simeq \mathrm{Hom}_{M_{\psi_A}}(\pi_{N,\psi_A}, \rho).$$

Using the multiplicity one theorem for $\mathrm{GL}(n)$ (see Theorem 2.4), we conclude that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{M_{\psi_A}}(\pi_{N,\psi_A}, \rho) = 1$$

and it follows that

$$\pi_{N,\psi_A} \simeq \rho$$

as M_{ψ_A} representations. □

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